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Resonance Eigenfunctions of a Dilation-Analytic Schrödinger Operator, Based on the Mellin Transform

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We consider a dilation-analytic Schrödinger operator represented (by the Mellin transform) in the space $\mathscr{H}_M := \{f: \mathbb{R} \to \mathscr{h} \mid f$ is measurable and $\int_{-\infty}^{\infty} \|f(\lambda)\|_{\mathscr{h}}^2 \, d\lambda < \infty\}$, $\mathscr{h}$ is $L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$. In this representation a notion of resonance eigenfucntions is defined by using a certain Gelfand triple. We find an isomorphic connection between the space of resonance eigenfunctions and the space $N(H_M(\theta) - z_0)$, $\operatorname{Im} \theta > -\frac{1}{2} \operatorname{Arg} z_0$, where $N(H_M(\theta) - z_0)$ is the space of eigenfunctions associated with a resonance z_0 and the θ-dilated operator $H_M(\theta)$ in the space $\mathscr{H}_M$. © 1986 Academic Press, Inc.

INTRODUCTION

Our purpose is to introduce the notion of resonance eigenfunctions for a dilation-analytic Schrödinger operator. This has been done for a multiplicative, radial, and analytic potential [2, p. 339], but not for our more general class of potentials.

The starting point is the space of eigenfunctions $N(H(\theta) - z_0)$ associated with a resonance z_0 and the θ-dilated operator $H(\theta)$ in the position space representation, $\operatorname{Im} \theta > -\frac{1}{2} \operatorname{Arg} z_0$.

In the case of a multiplicative, radial, and analytic potential, the (in relation to our work basic) observations are:

(1) There exists a basis $\{\psi_i(\theta)\}_{i=1}^n$ (n finite) of $N(H(\theta) - z_0)$ for $\operatorname{Im} \theta > -\frac{1}{2} \operatorname{Arg} z_0$, such that $\psi_i(\theta)$ is analytic in θ and (explicitly)

(2) $\psi_i(\theta)$ is of the form $\psi_i(\theta)(r) = \varphi_i(re^{i\theta})$, where $\{\varphi_i(r)\}_{i=1}^n$ is a basis of the space of resonance eigenfunctions. We remark that $\varphi_i(r)$ is analytic in r.

(3) Furthermore we have for $\operatorname{Im} \theta \geqslant 0$ that $\varphi_i(re^{i\theta}) \sim e^{i(z_0)^{1/2} re^{i\theta}}$ for $r \to \infty$.

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For the connection between the space of resonance eigenfunctions and an analytically continued resolvent (associated with the Schrödinger operator) we refer to [2, p. 344, (12.40)].

We find the problem of generalizing the notion of resonance eigenfunctions to our more general case interesting (in view of the above connection), and we are motivated by the above observations. For instance we expect that in a suitable space (for a multiplicative, radial, and analytic potential, this space is a space of at most exponentially increasing functions), in which the representation space ($L^2(\mathbb{R}^3)$) is continuously and densely embedded, we can continue any given vector belonging to $N(H(\theta)-z_0)$ ($\operatorname{Im}\theta > -\frac{1}{2}\operatorname{Arg} z_0$) analytically in θ to $\theta = 0$. We expect that this continuation does not belong to the representation space, whenever $-\frac{1}{2}\operatorname{Arg} z_0 \geqslant \operatorname{Im}\theta \geqslant 0$ and the continuation is not identical to zero (this is the case for the functions $\varphi_i(re^{i\theta})$ in (3)).

In the attempt to carry out the above program we meet technical problems and we will not proceed in this way (Is the program feasible?). What we actually shall do is to exchange the representation space $L^2(\mathbb{R}^3)$ for the Mellin transformed space $\mathscr{H}_M = \{f \colon \mathbb{R} \to \mathscr{k} \mid f$ is measurable and $\int_{-\infty}^{\infty} \|f(\lambda)\|_{\mathscr{k}}^2 \, d\lambda < \infty\}$. In this representation space we shall make the above continuation, which turns out to be extremely simple; in fact it is given by multiplication with the function $e^{i\theta(\cdot)}$ (according to (8)) with a function, which turn out to be what we call a resonance eigenfunction.

Using a certain Gelfand triple ((1)) we define the space of resonance eigenfunctions (Definition 11) by means of the operators $H_{M,-a}$ and $R_{M,a^+,-a}(z)$ (Diagrams 3 and 2, respectively).

Our results are collected in Theorem 12 and Theorem 15.

Definitions and Notation

Given Hilbertspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ we denote by $B(\mathscr{H}_1\ \mathscr{H}_2)$, $C(\mathscr{H}_1, \mathscr{H}_2)$, and $O(\mathscr{H}_1, \mathscr{H}_2)$ respectively the space of all bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$, the subspace of compact operators from $\mathscr{H}_1$ to $\mathscr{H}_2$, and the space of all linear operators with domain space $\mathscr{H}_1$ and range space $\mathscr{H}_2$.

If $\mathscr{H}_1 = \mathscr{H}_2$ we set

$$B(\mathscr{H}_1) = B(\mathscr{H}_1, \mathscr{H}_2)$$

$$C(\mathscr{H}_1) = C(\mathscr{H}_1, \mathscr{H}_2)$$

and

$$O(\mathscr{H}_1) = O(\mathscr{H}_1, \mathscr{H}_2).$$

Given closed subspaces $\mathscr{H}'_1 \subset \mathscr{H}_1$, $\mathscr{H}'_2 \subset \mathscr{H}_2$, and $H \in O(\mathscr{H}_1, \mathscr{H}_2)$ it is convenient to denote by $H_{\| B(\mathscr{H}'_1 \mathscr{H}'_2)}$ the operator defined by

$$\mathscr{D}(H_{\| B(\mathscr{H}'_1 \mathscr{H}'_2)}) = \{ f \in \mathscr{D}(H) \cap \mathscr{H}'_1 \mid Hf \in \mathscr{H}'_2 \}$$

and

$$H_{\| B(\mathscr{H}'_1 \mathscr{H}'_2)} f = Hf, \qquad \text{for} \quad f \in \mathscr{D}(H_{\| B(\mathscr{H}'_1 \mathscr{H}'_2)}).$$

We require that this operator belongs to $B(\mathscr{H}'_1, \mathscr{H}'_2)$.

The operators $H_{\| C(\mathscr{H}'_1 \mathscr{H}'_2)}$ and $H_{\| O(\mathscr{H}'_1 \mathscr{H}'_2)}$ are defined in a similarly way.

The graph of an operator A is denoted by $\Gamma(A)$.

By $\mathbb{R}$, $\mathbb{R}_-$, $\bar{\mathbb{R}}_+$, and $\mathbb{N}$ we shall mean $(-\infty, \infty)$, $(-\infty, 0)$, $[0, \infty)$, and $\{1, 2, \ldots\}$, respectively.

We consider the dilation group $\{U(\theta)\}_{\theta \in \mathbb{R}}$ acting on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3)$. Setting $A = \frac{1}{2}(XP + PX)$, the generator of $U(\theta)$, we have for $f \in \mathscr{H}$, $\theta \in \mathbb{R}$, and $x \in \mathbb{R}^3$ that $(U(\theta)f)(x) = (e^{i\theta A} f)(x) = e^{3/2\theta} f(e^\theta x)$.

The spectral family of A is denoted by $\{P_\lambda\}_{\lambda \in \mathbb{R}}$.

For some $a \in \mathbb{R}$, satisfying $0 < a < \pi/2$, we consider the operator $H = H_0 + V$ on $\mathscr{H}$, where $H_0 = -\Delta$ and V satisfies the following three conditions:

(1) V is symmetric and H_0-compact.

(2) V is a-dilation-analytic; that is, $V(\theta) := U(\theta)\, VU(-\theta)$, $\theta \in \mathbb{R}$, has an analytic extension in θ from $\theta \in \mathbb{R}$ to $\{\theta_1 \mid |\mathrm{Im}\, \theta_1| < a\}$ as a $B(\mathscr{D}(H_0), \mathscr{H})$-valued function (denoted by $V(\theta)$).

(3) $V(\theta)$ has a continuous extension from $\{\theta_1 \mid |\mathrm{Im}\, \theta_1| < a\}$ to $\{\theta_1 \mid |\mathrm{Im}\, \theta_1| \leqslant a\}$ as a $B(\mathscr{D}(H_0), \mathscr{H})$-valued function.

DEFINITION. By the condition $V(\theta) = U(\theta)\, VU(-\theta)$, $\theta \in \mathbb{R}$, has an analytic extension in θ from $\theta \in \mathbb{R}$ to $\{\theta_1 \mid |\mathrm{Im}\, \theta_1| \leqslant a\}$ as a $B(\mathscr{D}(H_0), \mathscr{H})$-valued function, we shall mean the above conditions (2) and (3).

Throughout the rest of this paper we shall always mean analytic in the interior of Q and continuous on Q, when we write that a certain function is analytic in a set Q.

Setting $H_0(\theta) = e^{-2\theta} H_0$ for $\theta \in \{\theta_1 \mid |\mathrm{Im}\, \theta_1| \leqslant a\}$, we denote by $H(\theta)$ the operator $H_0(\theta) + V(\theta)$, $\theta \in \{\theta_1 \mid |\mathrm{Im}\, \theta_1| \leqslant a\}$.

Clearly, $H(\theta)$ is an analytic extension of $U(\theta)\, HU(-\theta)$, $\theta \in \mathbb{R}$, in θ from $\theta \in \mathbb{R}$ to $\{\theta_1 \mid |\mathrm{Im}\, \theta_1| \leqslant a\}$ as a $B(\mathscr{D}(H_0), \mathscr{H})$-valued function.

DEFINITION. By the set of resonances we shall mean the set of isolated and nonreal eigenvalues of $H(ia)$.

According to the basic results concerning position of resonances, eigenvalues, etc., of a dilation-analytic Schrödinger operator (we refer to [3, 5, 6]), the set of resonances is confined to $\{z \neq 0 \mid 0 > \operatorname{Arg} z > -2a\}$.

Fixing b, $0 \leqslant b \leqslant a$, we consider the following sets:

$$S_{a^+} = \{\theta \mid 0 \leqslant \operatorname{Im} \theta \leqslant a\}$$

$$S_{b^-} = \{\theta \mid 0 \geqslant \operatorname{Im} \theta \geqslant -b\}$$

$$A_{a^+} = \{\varphi \in \mathscr{H} \mid \varphi(\theta) = U(\theta)\varphi \text{ has an analytic extension in } \theta \text{ from } \theta \in \mathbb{R} \text{ to } S_{a^+}\}$$

$$A_{b^-} = \{\varphi \in \mathscr{H} \mid \varphi(\theta) = U(\theta)\varphi \text{ has an analytic extension in } \theta \text{ from } \theta \in \mathbb{R} \text{ to } S_{b^-}\}$$

$$C^{++} = \{z \mid \operatorname{Im} z > 0 \text{ and } \operatorname{Re} z > 0\}$$

$$C_a = \{z \neq 0 \mid \pi/2 > \operatorname{Arg} z > -2a\}.$$

For $z \in C_a$

$$S_z = \{\theta \mid -\tfrac{1}{2} \operatorname{Arg} z < \operatorname{Im} \theta \leqslant a\} \cap S_{a^+}.$$

R_a is the set of eigenvalues of $H(ia)$ in $C_a \backslash C^{++}$.

$S(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.

$C_0(\mathbb{R})$ is the set of continuous functions on $\mathbb{R}$ vanishing at infinity.

$\mathscr{k}$ is $L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$.

The norm and the inner product on $\mathscr{k}$ are denoted by $\|\cdot\|_{\mathscr{k}}$ and $(\cdot,\cdot)_{\mathscr{k}}$, respectively.

We define the following spaces of $\mathscr{k}$-valued, measurable functions on $\mathbb{R}$:

$$\mathscr{H}_M = \left\{ f \mid \|f\|^2_{\mathscr{H}_M} := \int_{-\infty}^{\infty} \|f(\lambda)\|^2_{\mathscr{k}}\, d\lambda < \infty \right\}$$

$$\mathscr{H}_{M,a^+} = \left\{ f \mid \|f\|^2_{\mathscr{H}_{M,a^+}} := \int_{-\infty}^{\infty} (1 + e^{-2a\lambda}) \cdot \|f(\lambda)\|^2_{\mathscr{k}}\, d\lambda < \infty \right\}$$

$$\mathscr{H}_{M,b^-} = \left\{ f \mid \|f\|^2_{\mathscr{H}_{M,b^-}} := \int_{-\infty}^{\infty} (1 + e^{2b\lambda}) \cdot \|f(\lambda)\|^2_{\mathscr{k}}\, d\lambda < \infty \right\} \quad (0 \leqslant b \leqslant a)$$

$$\mathscr{H}_{M,-b} = \left\{ f \mid \|f\|^2_{\mathscr{H}_{M,-b}} := \int_{-\infty}^{\infty} (1 + e^{2b\lambda})^{-1} \cdot \|f(\lambda)\|^2_{\mathscr{k}}\, d\lambda < \infty \right\} \quad (0 \leqslant b \leqslant a).$$

The norm on $\mathscr{H}$ is denoted by $\|\cdot\|$, and as indicated above the canonical norms on $\mathscr{H}_M$, $\mathscr{H}_{M,a^+}$, $\mathscr{H}_{M,b^-}$, and $\mathscr{H}_{M,-b}$ are denoted by $\|\cdot\|_{\mathscr{H}_M}$, $\|\cdot\|_{\mathscr{H}_{M,a^+}}$, $\|\cdot\|_{\mathscr{H}_{M,b^-}}$, and $\|\cdot\|_{\mathscr{H}_{M,-b}}$, respectively. Similar notations are used for inner products.

Letting $v, w \in \mathscr{k}$, $(v, w)_{\mathscr{k}}$ is taken to be linear in v and antilinear in w, and similarly for all other inner products.

For $f \in \mathscr{H}_{M,b^-}$ and $g \in \mathscr{H}_{M,-b}$ we define

$$\langle f, g \rangle_{b^-,-b} = \int_{-\infty}^{\infty} (f(\lambda), g(\lambda))_{\mathscr{k}}\, d\lambda. \tag{1}$$

Clearly $\mathscr{H}_{M,b^-} \subset \mathscr{H}_M \subset \mathscr{H}_{M,-b}$ and $(\mathscr{H}_{M,b^-})^* = \mathscr{H}_{M,-b}$ with respect to the form defined above.

We use the fact that the Mellin transform M is *a* unitary operator from $\mathscr{H}$ to $\mathscr{H}_M$ and A is diagonalized by M [1]. (Other facts about the Mellin transform will not be utilized.)

Soon we shall show that $MA_{b^-} = \mathscr{H}_{M,b^-}$.

Fixing $\theta \in \{\theta_1 \mid |\operatorname{Im}\theta_1| \leqslant a\}$ we define $H_M(\theta) = MH(\theta)M^{-1}$.

Fixing $z \in C_a \backslash R_a$ and $\theta \in S_z$ we define

$$R_M(\theta, z) = MR(\theta, z)M^{-1} = M(H(\theta) - z)^{-1}M^{-1}.$$

The operators above are illustrated in Diagram 1 below. For later use Diagrams 2 and 3 are given too.

Diagram 1. $z \in C_a \backslash R_a$ and $\theta \in S_z$,

$$\begin{tikzcd}
\mathscr{H}_M \arrow[d, bend left, "R_M(\theta, z)"] & & \mathscr{H} \arrow[ll, "M"'] \arrow[d, bend left, "R(\theta, z)"] \\
\mathscr{H}_M \arrow[u, bend left, "H_M(\theta)"] & & \mathscr{H} \arrow[ll, "M"] \arrow[u, bend left, "H(\theta)"]
\end{tikzcd}$$

Diagram 2. $z \in C_a \backslash R_a$,

$$\begin{tikzcd}
\mathscr{H}_{M,a^+} \subset \mathscr{H}_M \subset \mathscr{H}_{M,-a} \arrow[from=1-1, to=1-1, phantom]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{H}_{M,a^+} \arrow[rr, bend right, "R_{M,a^+,-a}(z)"'] & \subset \mathscr{H}_M \subset & \mathscr{H}_{M,-a}
\end{tikzcd}$$

Diagram 3. $\theta \in S_{a^+}$ and $b = a - \operatorname{Im}\theta$,

$$\begin{tikzcd}
\mathscr{H}_{M,b^-} & A_{b^-} \arrow[l, "M"'] \\
\mathscr{H}_{M,b^-} \arrow[u, bend left, "H_{M,b^-}(\bar\theta)"] & A_{b^-} \arrow[l, "M"'] \arrow[u, bend left, "H_{b^-}(\bar\theta)"] \\
\mathscr{H}_{M,-b} \arrow[r, "H_{M,-b}(\theta)"] & \mathscr{H}_{M,-b}
\end{tikzcd}$$

$H_{M,b^-}(\bar{\theta})$ in Diagram 3 above is $H_M(\bar{\theta})_{\| O(\mathscr{H}_{M,b^-})}$ and $H_{M,-b}(\theta)$ is defined to be $(H_{M,b^-}(\bar{\theta}))^*$ by the duality (1). We emphasize the connection between θ and b. We shall later show (Lemma 5) that $H_{M,-b}(\theta)$ in this way is well defined. $H_{b^-}(\bar{\theta})$ is defined to be $M^{-1}H_{M,b^-}(\bar{\theta})M$.

The operators $H_M(0)$, $H_{M,a^-}(0)$, etc., will be abbreviated as H_M, H_{M,a^-}, etc.

We now explain Diagram 2: For $z \in C^{++}$ and $\theta \in \mathbb{R}$, $R_M(\theta, z) = e^{i\theta(\cdot)}R_M(z)\, e^{-i\theta(\cdot)}$ ($R_M(z) = R_M(0, z)$). Hence, $R_M(z) = e^{-i\theta(\cdot)}R_M(\theta, z)\, e^{i\theta(\cdot)}$.

By analytic extension in θ to $\theta = ia \in S_{a^+}$ (see the remark below), we get

$$R_M(z)_{\| B(\mathscr{H}_{M,a^+}, \mathscr{H}_{M,-a})} = e^{a(\cdot)}R_M(ia, z)\, e^{-a(\cdot)}. \tag{2}$$

Remark that we interpret $e^{i\theta(\cdot)}$ as an analytic $B(\mathscr{H}_{M,a^+}, \mathscr{H}_M)$-valued function in $\theta \in S_{a^+}$ and $e^{-i\theta(\cdot)}$ as an analytic $B(\mathscr{H}_M, \mathscr{H}_{M,-a})$-valued function in $\theta \in S_{a^+}$. Furthermore we use the fact that $R_M(\theta, z)$ is analytic in $\theta \in S_{a^+}$ as a $B(\mathscr{H}_M)$-valued function.

Clearly the right-hand side of (2) has a meromorphic continuation in z from $z \in C^{++}$ to C_a with poles at R_a, given by $e^{a(\cdot)}R_M(ia, z)\, e^{-a(\cdot)}$. This continuation is denoted by $R_{M,a^+,-a}(z)$.

Hence we have proved that $R_M(z)_{\| B(\mathscr{H}_{M,a^+}, \mathscr{H}_{M,-a})}$ for $z \in C^{++}$ has a meromorphic continuation in z to C_a with poles at R_a given by

$$R_{M,a^+,-a}(z) = e^{a(\cdot)}R_M(ia, z)\, e^{-a(\cdot)}. \tag{3}$$

1. Resonance Eigenfunctions

We prove our main result (Theorem 12) by collecting results from a series of lemmas.

LEMMA 1. *Fixing b, $0 \leqslant b \leqslant a$, we have equipping the space A_{b^-} with the norm $\| \varphi \|_{A_{b^-}} = (\| \varphi \|^2 + \| \varphi(-ib) \|^2)^{1/2}$ ($\varphi \in A_{b^-}$) that M is a unitary operator from A_{b^-} onto $\mathscr{H}_{M,b^-}$.*

Proof. The Lemma is easily proved from the fact that M diagonalizes the generator A (see also [7, pp. 641–642]).

We shall now study the operators $H_{M,b^-}(\bar{\theta})$ (see Diagram 3) in some detail. For convenience we consider the conjugated operators $H_{b^-}(\bar{\theta}) = M^{-1}H_{M,b^-}(\bar{\theta})M$ (Diagram 3). By Lemma 1, $H_{b^-}(\bar{\theta}) = H(\bar{\theta})_{\| O(A_{b^-})}$.

LEMMA 2. *For $z \in \mathbb{R}_-$ and $\theta \in \{\theta_1 \mid \operatorname{Im} \theta_1 = a - b\}$, the following sets are equal and independent of $z \in \mathbb{R}_-$ and $\theta \in \{\theta_1 \mid \operatorname{Im} \theta_1 = a - b\}$:*

(1) $\mathscr{D}(H_{b^-}(\bar{\theta})) = \{\psi \in A_{b^-} \mid \psi \in \mathscr{D}(H_0)$ *and* $H(\bar{\theta})\psi \in A_{b^-}\}$.

(2) $O_z := \{\psi \in A_{b^-} \mid \exists \varphi \in A_{b^-} : \psi = (H_0 - z)^{-1}\varphi\}$.

(3) $\{\psi \in A_{b^-} \mid \psi(\theta) \in \mathscr{D}(H_0)$ *for all* $\theta \in S_{b^-}$, *and* $\psi(\theta)$ *is analytic as a* $\mathscr{D}(H_0)$*-valued function*$\}$.

Proof. The first resolvent equation provides easily that for all $z \in \mathbb{R}_-$ $O_z = \{\psi \in A_{b^-} \mid \exists \varphi \in A_{b^-} : \psi = (H_0 + 1)^{-1}\varphi\}$.

(3) $\subset$ (1) and (2) $\subset$ (3) are easily proved too.

We now prove that (1) $\subset$ (2) for one $z \in \mathbb{R}_-$: Let $\psi \in \mathscr{D}(H_{b^-}(\bar{\theta}))$ and choose $z \in \mathbb{R}_- \backslash \sigma(H(\bar{\theta}))$. Let $\varphi_1 = (H(\bar{\theta}) - z)\psi$. Then $\varphi_1 \in A_{b^-}$. Using the second resolvent equation, we have

$$\begin{aligned}\psi &= (H(\bar{\theta}) - z)^{-1}\varphi_1 \\ &= [(H_0 - z)^{-1} - (H_0 - z)^{-1}(H(\bar{\theta}) - H_0)(H(\bar{\theta}) - z)^{-1}]\varphi_1 \\ &= (H_0 - z)^{-1}[1 - (H(\bar{\theta}) - H_0)(H(\bar{\theta}) - z)^{-1}]\varphi_1.\end{aligned}$$

Because $[1 - (H(\bar{\theta}) - H_0)(H(\bar{\theta}) - z)^{-1}]\varphi_1$ belongs to A_{b^-}, we have shown that $\psi = (H_0 - z)^{-1}\varphi$, where $\varphi = [1 - (H(\bar{\theta}) - H_0)(H(\bar{\theta}) - z)^{-1}]$ $\varphi_1 \in A_{b^-}$.

Hence $\psi \in O_z$, and the Lemma is proved.

Remark 3. To be used in the proof of Lemma 5, we remark that $\{\psi \in A_{b^-} \mid \exists \varphi \in A_{b^-} : \psi = (H_0 + 1)^{-1}\varphi\}$ is dense in $\mathscr{H}$. This is easily proved from the fact that $\mathscr{D}(H_0)$ and A_{b^-} are dense in $\mathscr{H}$.

The following Lemma will be used in the proof of Lemma 5.

For $\eta \in S(\mathbb{R})$ we denote by $F\eta = \hat{\eta}$ and $F^{-1}\eta = \check{\eta}$ the Fourier-transformation and the inverse Fourier-transformation of η, respectively.

LEMMA 4. *For* $\eta \in B := \{\eta_1 \in S(\mathbb{R}) \mid F\eta_1 \in C_0^\infty(\mathbb{R})\}$, we have the identity

$$(H_0 + 1)\,\eta(A)(H_0 + 1)^{-1} = \eta(A)(H_0 + 1)^{-1} + (\widecheck{e^{2(\cdot)}\hat{\eta}})(A)\,H_0(H_0 + 1)^{-1},$$

where $e^{2(\cdot)}: \mathbb{R} \to \mathbb{R}$ *is given by* $s \to e^{2s}$.

Proof. For $C = (1/2\pi)^{1/2}$, $\eta \in B, f \in \mathscr{H}$, and $g \in \mathscr{D}(H_0)$ we consider

$$\begin{aligned}&(\eta(A)(H_0 + 1)^{-1}f, (H_0 + 1)\,g) \\ &\quad = \int_{-\infty}^{\infty} \eta(t)\,d(P_t(H_0 + 1)^{-1}f, (H_0 + 1)\,g) \\ &\quad = C\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\hat{\eta}(s)\,e^{ist}\,ds\right)d(P_t(H_0 + 1)^{-1}f, (H_0 + 1)\,g) \\ &\quad = C\int_{-\infty}^{\infty} ds\hat{\eta}(s)\int_{-\infty}^{\infty} e^{ist}\,d(P_t(H_0 + 1)^{-1}f, (H_0 + 1)\,g) \qquad \text{(Fubini)}\end{aligned}$$

$$= C \int_{-\infty}^{\infty} ds\hat{\eta}(s)(e^{isA}(H_0+1)^{-1}f, (H_0+1)g)$$

$$= C \int_{-\infty}^{\infty} ds\hat{\eta}(s)((H_0+1)\, e^{isA}(H_0+1)^{-1}f, g)$$

$$= C \int_{-\infty}^{\infty} ds\hat{\eta}(s)(e^{isA}(e^{-isA}H_0 e^{isA}+1)(H_0+1)^{-1}f, g)$$

$$= C \int_{-\infty}^{\infty} ds\hat{\eta}(s)(e^{isA}(e^{2s}H_0+1)(H_0+1)^{-1}f, g)$$

$$= C \int_{-\infty}^{\infty} ds\hat{\eta}(s) \int_{-\infty}^{\infty} e^{ist}\, d(P_t(e^{2s}H_0+1)(H_0+1)^{-1}f, g)$$

$$= \int_{-\infty}^{\infty} \widecheck{(e^{2(\cdot)}\hat{\eta})}(t)\, d(P_t H_0(H_0+1)^{-1}f, g)$$

$$+ \int_{-\infty}^{\infty} \eta(t)\, d(P_t(H_0+1)^{-1}f, g) \qquad \text{(Fubini)}$$

$$= ([\widecheck{(e^{2(\cdot)}\hat{\eta})}(A)H_0 + \eta(A)](H_0+1)^{-1}f, g).$$

We have proved that for $\eta \in B$ and all $f \in \mathscr{H}$ and all $g \in \mathscr{D}(H_0)$

$$(\eta(A)(H_0+1)^{-1}f, (H_0+1)g)$$
$$= ([\widecheck{(e^{2(\cdot)}\hat{\eta})}(A)H_0 + \eta(A)](H_0+1)^{-1}f, g).$$

Hence for all $f \in \mathscr{H}$, $\eta(A)(H_0+1)^{-1} f \in \mathscr{D}(H_0)$ and

$$(H_0+1)\,\eta(A)(H_0+1)^{-1}f = [\widecheck{(e^{2(\cdot)}\hat{\eta})}(A)H_0 + \eta(A)](H_0+1)^{-1}f.$$

The identity is proved.

LEMMA 5. *$\mathscr{D}(H_{b^-}(\bar{\theta})) = \{\psi \in A_{b^-} \mid \exists \varphi \in A_{b^-} : \psi = (H_0+1)^{-1}\varphi\}$ is dense in A_{b^-} with respect to the $\|\cdot\|_{A_{b^-}}$-topology.*

Proof. Using Lemma 4 we get for $\varphi_1 \in A_{b^-}$ and $\eta \in B$ that

$$\eta(A)(H_0+1)^{-1}\varphi_1 = (H_0+1)^{-1}[\eta(A) + \widecheck{(e^{2(\cdot)}\hat{\eta})}(A)H_0](H_0+1)^{-1}\varphi_1.$$

Setting $\varphi = [\eta(A) + \widecheck{(e^{2(\cdot)}\hat{\eta})}(A)H_0](H_0+1)^{-1}\varphi_1$, we find that $\varphi \in A_{b^-}$ and

$$\eta(A)(H_0+1)^{-1}\varphi_1 = (H_0+1)^{-1}\varphi.$$

Hence, by Lemma 2, $\eta(A)(H_0+1)^{-1}\varphi_1 \in \mathscr{D}(H_{b^-}(\bar{\theta}))$ for all $\varphi_1 \in A_{b^-}$ and all $\eta \in B$.

It now suffices to prove that given $\psi \in A_{b^-}$, which satisfies the condition: For all $\eta \in B$ and for all $\varphi_1 \in A_{b^-}$, $(\eta(A)(H_0+1)^{-1}\varphi_1, \psi)_{A_{b^-}} = 0$; then $\psi = 0$.

Assuming that for all $\eta \in B$ and all $\varphi_1 \in A_{b^-}$, $(\eta(A)(H_0+1)^{-1}\varphi_1, \psi)_{A_{b^-}} = 0$, we consider

$$
\begin{aligned}
&(\eta(A)(H_0+1)^{-1}\varphi_1, \psi)_{A_{b^-}} \\
&= \int_{-\infty}^{\infty} (1+e^{2b\lambda})((M\eta(A)(H_0+1)^{-1}\varphi_1)(\lambda), \quad (M\psi)(\lambda))_{\mathscr{h}}\, d\lambda \\
&= \int_{-\infty}^{\infty} (1+e^{2b\lambda})\, \eta(\lambda)((M(H_0+1)^{-1}\varphi_1)(\lambda), (M\psi)(\lambda))_{\mathscr{h}}\, d\lambda \\
&= C \int_{-\infty}^{\infty} (1+e^{2b\lambda}) \left(\int_{-\infty}^{\infty} dt e^{i\lambda t} \hat{\eta}(t) \right) \\
&\quad \times ((M(H_0+1)^{-1}\varphi_1)(\lambda), (M\psi)(\lambda))_{\mathscr{h}}\, d\lambda \qquad \left(C = \left(\frac{1}{2\pi}\right)^{1/2} \right).
\end{aligned}
$$

By setting $g(\lambda) = ((1+e^{2b\lambda})^{1/2}(M(H_0+1)^{-1}\varphi_1)(\lambda), (1+e^{2b\lambda})^{1/2}(M\psi))_{\mathscr{h}}$, we find that $g \in L^1(\mathbb{R})$ and

$$
(\eta(A)(H_0+1)^{-1}\varphi_1, \psi)_{A_{b^-}} = C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} dt \hat{\eta}(t)\, e^{i\lambda t} \right) g(\lambda)\, d\lambda.
$$

The left-hand side is equal to zero by assumption, and by Fubini the right-hand side is equal to $\int_{-\infty}^{\infty} \hat{\eta}(t)\, \check{g}(t)\, dt$. That is, for all $\eta \in B$

$$
\int_{-\infty}^{\infty} \hat{\eta}(t)\, \check{g}(t)\, dt = 0.
$$

By the Du Bois–Reymond theorem we find that $\check{g}(t) = 0$ a.e. Because F^{-1}: $L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is one-to-one, $g(t) = 0$, a.e. Hence, $((M(H_0+1)^{-1}\varphi_1)(t), (M\psi)(t))_{\mathscr{h}} = 0$ a.e. and thus we get for all $\varphi_1 \in A_{b^-}$ that $((H_0+1)^{-1}\varphi_1, \psi) = 0$.

We use Remark 3, and conclude $\psi = 0$.

The Lemma is proved.

Because of Lemma 5 the operators $H_{M,-b}(\theta)$ in Diagram 3 are well-defined.

LEMMA 6. *$\Gamma(H_{b^-}(\bar{\theta}))$ is dense in $\Gamma(H(\bar{\theta}))$.*

Proof. Let $x \in \Gamma(H(\bar{\theta}))$. Then for some $\varphi \in \mathscr{H}$, $x = \{(H_0+1)^{-1}\varphi, H(\bar{\theta})(H_0+1)^{-1}\varphi\}$.

Because A_{b^-} is dense in $\mathscr{H}$, we can find $\{\varphi_k\}_1^\infty \subset A_{b^-}$ such that $\varphi_k \to \varphi$ in $\mathscr{H}$ for $k \to \infty$.

Then $\{\{(H_0+1)^{-1}\varphi_k, \ H(\bar{\theta})(H_0+1)^{-1}\varphi_k\}\}_1^\infty \subset \Gamma(H_{b^-}(\bar{\theta}))$ and $\{(H_0+1^{-1}\varphi_k, \ H(\bar{\theta})(H_0+1)^{-1}\varphi_k\} \to \{(H_0+1)^{-1}\varphi, \ H(\bar{\theta})(H_0+1)^{-1}\varphi\}$ for $k \to \infty$.

Hence the Lemma is proved.

Before stating the next lemma we remark that for $\theta \in S_{a^+}$

$$(H_M(\bar{\theta}))^* = M(H(\bar{\theta}))^* M^{-1} = MH(\theta) M^{-1} = H_M(\theta).$$

LEMMA 7. $H_{M,-b}(\theta)_{\| O(\mathscr{H}_M)} = H_M(\theta)$. *In particular we have for* $\theta = 0$

$$H_{M,-a \| O(\mathscr{H}_M)} = H_M.$$

Proof. First we prove that $H_{M,-b}(\theta)_{\| O(\mathscr{H}_M)} \subset H_M(\theta)$:

Let $g \in \mathscr{D}(H_{M,-b}(\theta)) \cap \mathscr{H}_M$ and assume $H_{M,-b}(\theta)\, g \in \mathscr{H}_M$. We shall show that $g \in \mathscr{D}(H_M(\theta))$ and $H_M(\theta)\, g = H_{M,-b}(\theta)\, g$.

For all $f \in \mathscr{D}(H_{M,b^-}(\bar{\theta}))$

$$\langle H_{M,b^-}(\bar{\theta})\, f, g \rangle_{b^-,-b} = \langle f, H_{M,-b}(\theta)\, g \rangle_{b^-,-b}. \tag{4}$$

Because g, $H_{M,-b}(\theta)\, g \in \mathscr{H}_M$, we have for all $f \in \mathscr{D}(H_{M,b^-}(\bar{\theta}))$

$$(H_M(\bar{\theta})\, f, g)_{\mathscr{H}_M} = (f, H_{M,-b}(\theta)\, g)_{\mathscr{H}_M}. \tag{5}$$

From this equation and Lemma 6 we get for all $f \in \mathscr{D}(H_M(\bar{\theta}))$

$$(H_M(\bar{\theta})\, f, g)_{\mathscr{H}_M} = (f, H_{M,-b}(\theta)\, g)_{\mathscr{H}_M}.$$

We now conclude that $g \in \mathscr{D}(H_M(\theta))$ and $H_M(\theta)\, g = H_{M,-b}(\theta)\, g$. Hence $H_{M,-b}(\theta)_{\| O(\mathscr{H}_M)} \subset H_M(\theta)$.

We now prove that $H_{M,-b}(\theta)_{\| O(\mathscr{H}_M)} \supset H_M(\theta)$:

Let $g \in \mathscr{D}(H_M(\theta))$ be given, then we shall show that $g \in \mathscr{D}(H_{M,-b}(\theta))$ and $H_{M,-b}(\theta)\, g = H_M(\theta)\, g$.

For all $f \in \mathscr{D}(H_M(\bar{\theta}))$

$$(H_M(\bar{\theta})\, f, g)_{\mathscr{H}_M} = (f, H_M(\theta)\, g)_{\mathscr{H}_M}.$$

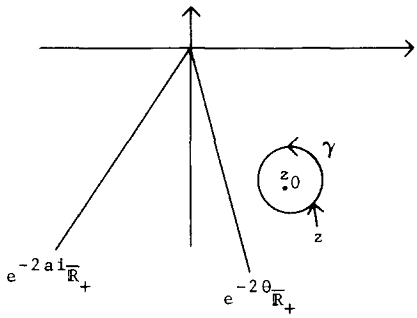
In particular, for all $f \in \mathscr{D}(H_{M,b^-}(\bar{\theta}))$

$$\langle H_{M,b^-}(\bar{\theta})\, f, g \rangle_{b^-,-b} = \langle f, H_M(\theta)\, g \rangle_{b^-,-b}.$$

Hence $g \in \mathscr{D}(H_{M,-b}(\theta))$ and $H_{M,-b}(\theta)\, g = H_M(\theta)\, g$. The Lemma is proved.

We now consider a resonance $z_0 \in R_a$, $\theta \in S_{z_0}$ and a path γ as in the diagram below.

Diagram 4.



The following two facts are well known [4, Lemma 4.8]:

(1) $N(H(\theta)-z_0)$ (the eigenspace of $H(\theta)$ associated with z_0) is finite dimensional.

(2) There exists a basis $\{\psi_i(\theta)\}_i$ of $N(H(\theta)-z_0)$, such that $\psi_i(\theta)$ is analytic in $\theta \in S_{z_0}$ and for all $\theta_1 \in \mathbb{R}$, $\theta \in S_{z_0}$

$$U(\theta_1)\,\psi_i(\theta) = \psi_i(\theta_1 + \theta).$$

Explicitly, given $\psi(\theta) \in N(H(\theta)-z_0)$ and analytic (as above) in $\theta \in S_{z_0}$, then for some $\varphi \in A_{a^+}$

$$\psi(\theta) = -\frac{1}{2\pi i}\int_\gamma R(\theta, z)\,\varphi(\theta)\,dz. \tag{6}$$

We fix $\psi(\theta)$ and φ as above. For $(\theta - ia) \in \mathbb{R}$ and $z \in \gamma$

$$\begin{aligned} MR(\theta, z)\,\varphi(\theta) &= MU(\theta - ia)\,R(ia, z)\,U(-(\theta - ia))\,U(\theta - ia)\,\varphi(ia) \\ &= e^{i(\theta - ia)(\cdot)}MR(ia, z)\,\varphi(ia). \end{aligned}$$

Thus, clearly $MR(\theta, z)\,\varphi(\theta)$ has an analytic extension in θ from $(\theta - ia) \in \mathbb{R}$ to $\theta \in S_{a^+}$ as an $\mathscr{H}_{M,-a}$-valued function given by $e^{i(\theta - ia)(\cdot)}MR(ia, z)\,\varphi(ia)$.

For $\theta \in S_{z_0}$ an analytic extension is given by $MR(\theta, z)\,\varphi(\theta) \in \mathscr{H}_M \subset \mathscr{H}_{M,-a}$.

Hence we have proved (by uniqueness of analytic extension) that the $\mathscr{H}_M$-valued analytic function $MR(\theta, z)\,\varphi(\theta)$ defined for $\theta \in S_{z_0}$ has an analytic continuation in θ to $\theta \in S_{a^+}$ as an $\mathscr{H}_{M,-a}$-valued function given by

$$[MR(\cdot, z)\,\varphi(\cdot)]^\sim(\theta) = e^{i(\theta - ia)(\cdot)}MR(ia, z)\,\varphi(ia). \tag{7}$$

Moreover for $\theta \in S_{a^+}$ the continuation above belongs to $\mathscr{H}_{M,-b}$ $(b = a - \operatorname{Im} \theta)$.

Using (7) we now easily find that the function $M\psi(\theta)$ defined for $\theta \in S_{z_0}$ has an analytic continuation in θ as an $\mathscr{H}_{M,-a}$-valued function to $\theta \in S_{a^+}$ given by

$$[M\psi]^{\sim}(\theta) = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} MR(ia, z)\, \varphi(ia)\, dz. \tag{8}$$

For $\theta \in S_{a^+}$, $[M\psi]^{\sim}(\theta) \in \mathscr{H}_{M,-b}$ $(b = a - \operatorname{Im} \theta)$.

We now prove that $[M\psi]^{\sim}(\theta) \in \mathscr{D}(H_{M,-b}(\theta))$ and $(H_{M,-b}(\theta) - z_0)$ $[M\psi]^{\sim}(\theta) = 0$.

LEMMA 8. Fixing $\theta \in S_{a^+}$ and $z \in \gamma$ we have

(1) $e^{i(\theta - ia)(\cdot)} MR(ia, z)\, \varphi(ia) \in \mathscr{D}(H_{M,-b}(\theta))$

and

$$\begin{aligned} H_{M,-b}(\theta)&[e^{i(\theta - ia)(\cdot)} MR(ia, z)\, \varphi(ia)] \\ &= e^{i(\theta - ia)(\cdot)} M(zR(ia, z) + I)\, \varphi(ia); \end{aligned}$$

(2) $[M\psi]^{\sim}(\theta) \in \mathscr{D}(H_{M,-b}(\theta))$

and

$$(H_{M,-b}(\theta) - z_0)[M\psi]^{\sim}(\theta) = 0.$$

In particular for $\theta = 0$, $(H_{M,-a} - z_0)[M\psi]^{\sim}(0) = 0$.

Proof. Let $f \in \mathscr{D}(H_{M,b^-}(\bar{\theta}))$ and $(\theta_1 - ia) \in \mathbb{R}$. Then

$$\begin{aligned} &(e^{-i\overline{(\theta_1 - ia)}(\cdot)} H_M(\bar{\theta}_1) f, MR(ia, z)\, \varphi(ia))_{\mathscr{H}_M} \\ &= (MH(\overline{ia})\, M^{-1} e^{-i\overline{(\theta_1 - ia)}(\cdot)} f, MR(ia, z)\, \varphi(ia))_{\mathscr{H}_M} \\ &= (e^{-i\overline{(\theta_1 - ia)}(\cdot)} f, MH(ia)\, M^{-1} MR(ia, z)\, \varphi(ia))_{\mathscr{H}_M} \\ &= (f, e^{i(\theta_1 - ia)(\cdot)} M(zR(ia, z) + I)\, \varphi(ia))_{\mathscr{H}_M} \\ &= \langle f, e^{i(\theta_1 - ia)(\cdot)} M(zR(ia, z) + I)\, \varphi(ia)) \rangle_{b^-, -b}. \end{aligned}$$

Hence for $(\theta_1 - ia) \in \mathbb{R}$

$$\begin{aligned} &(e^{-i\overline{(\theta_1 - ia)}(\cdot)} H_M(\bar{\theta}_1) f, MR(ia, z)\, \varphi(ia))_{\mathscr{H}_M} \\ &\quad = \langle f, e^{i(\theta_1 - ia)(\cdot)} M(zR(ia, z) + I)\, \varphi(ia) \rangle_{b^-, -b}. \end{aligned} \tag{9}$$

We remark that $e^{-i(\overline{\theta_1 - ia})(\cdot)}H_M(\bar{\theta}_1)f$ has an analytic extension—given by the same expression—in $\bar{\theta}_1$ from $(\theta_1 - ia) \in \mathbb{R}$ to $\theta_1 \in S_\theta :=$ $\{\theta_2 \mid a \geqslant \operatorname{Im}\theta_2 \geqslant \operatorname{Im}\theta\}$. To see this we use the equation below for $(\theta_1 - ia) \in \mathbb{R}$:

$$e^{-i(\overline{\theta_1 - ia})(\cdot)}H_M(\bar{\theta}_1)f = MH(\overline{ia})(M^{-1}f)(-(\overline{\theta_1 - ia})).$$

For $n \in \mathbb{N}$ and $(\theta_1 - ia) \in \mathbb{R}$, we have

$$1_{(-n,n)}(\cdot)\, e^{-i(\overline{\theta_1 - ia})(\cdot)}H_M(\bar{\theta}_1)f = 1_{(-n,n)}(\cdot)\, MH(\overline{ia})(M^{-1}f)(-(\overline{\theta_1 - ia})).$$

By analytic extension in $\bar{\theta}_1$ to $\theta_1 \in S_\theta$ (the left-hand side clearly has such an extension, and the right-hand side too because of Lemma 2(3)), we get for $\theta_1 \in S_\theta$

$$\begin{aligned}&1_{(-n,n)}(\cdot)\, e^{-i(\overline{\theta_1 - ia})(\cdot)}H_M(\bar{\theta}_1)f\\ &\quad = 1_{(-n,n)}(\cdot)\, MH(\overline{ia})(M^{-1}f)(-(\overline{\theta_1 - ia})).\end{aligned}$$

Letting $n \to \infty$ we get for $\theta_1 \in S_\theta$ (remark that, $e^{-i(\overline{\theta_1 - ia})(\cdot)}H_M(\bar{\theta}_1)f \in \mathscr{H}_M$; this follows easily from the condition on f)

$$e^{-i(\overline{\theta_1 - ia})(\cdot)}H_M(\bar{\theta}_1)f = MH(\overline{ia})(M^{-1}f)(-(\overline{\theta_1 - ia})).$$

We conclude that the left-hand side is analytic in $\bar{\theta}_1$ $(\theta_1 \in S_\theta)$, because the right-hand side is (Lemma 2(3)).

Using the above remark, we extend (9) analytically in $\bar{\theta}_1$ to $\theta_1 = \theta \in S_\theta$ and get

$$\begin{aligned}&(e^{-i(\overline{\theta - ia})(\cdot)}H_M(\bar{\theta})f, MR(ia, z)\,\varphi(ia))_{\mathscr{H}_M}\\ &\quad = \langle f, e^{i(\theta - ia)(\cdot)}M(zR(ia, z) + I)\,\varphi(ia)\rangle_{b^-, -b}.\end{aligned}$$

Hence,

$$\begin{aligned}&\langle H_M(\bar{\theta})f, e^{i(\theta - ia)(\cdot)}MR(ia, z)\,\varphi(ia)\rangle_{b^-, -b}\\ &\quad = \langle f, e^{i(\theta - ia)(\cdot)}M(zR(ia, z) + I)\,\varphi(ia)\rangle_{b^-, -b}.\end{aligned} \tag{10}$$

Because (10) is true for all $f \in \mathscr{D}(H_{M,b^-}(\bar{\theta}))$, we have proved (1).

By integrating (10) we get for all $f \in \mathscr{D}(H_{M,b^-}(\bar{\theta}))$

$$\begin{aligned}&\left\langle H_{M,b^-}(\bar{\theta})f, -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)}MR(ia, z)\,\varphi(ia)\,dz\right\rangle_{b^-, -b}\\ &\quad = \left\langle f, -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)}M(zR(ia, z))\,\varphi(ia)\,dz\right\rangle_{b^-, -b}.\end{aligned}$$

Hence

$$[M\psi]^{\sim}(\theta) = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} MR(ia, z)\, \varphi(ia)\, dz \in \mathscr{D}(H_{M,-b}(\theta))$$

and

$$H_{M,-b}(\theta)[M\psi]^{\sim}(\theta) = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} M(zR(ia, z))\, \varphi(ia)\, dz.$$

Thus

$$\begin{aligned}
&(H_{M,-b}(\theta) - z_0)[M\psi]^{\sim}(\theta) \\
&\quad = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} M(zR(ia, z))\, \varphi(ia)\, dz \\
&\qquad - z_0\left(\frac{-1}{2\pi i}\right)\int_\gamma e^{i(\theta - ia)(\cdot)} MR(ia, z)\, \varphi(ia)\, dz \\
&\quad = e^{i(\theta - ia)(\cdot)} M\left(\frac{-1}{2\pi i}\right)\int_\gamma (z - z_0)\, R(ia, z)\, \varphi(ia)\, dz \\
&\quad = e^{i(\theta - ia)(\cdot)} M(H(ia) - z_0)\, \psi(ia) = 0.
\end{aligned}$$

Part (2) is proved.

LEMMA 9. *For $z \in C_a \backslash R_a$*

$$\mathscr{R}(R_{M,a^+,-a}(z)) \subset \mathscr{D}(H_{M,-a}) \qquad (\mathscr{R} \text{ for range}),$$

and for all $\varphi \in A_{a^+}$

$$(H_{M,-a} - z)\, R_{M,a^+,-a}(z)\, M\varphi = M\varphi.$$

Moreover, for all $\varphi \in A_{a^+}$

$$-\frac{1}{2\pi i}\int_\gamma R_{M,a^+,-a}(z)\, M\varphi \in \mathscr{D}(H_{M,-a}),$$

and

$$\begin{aligned}
&H_{M,-a}\left[-\frac{1}{2\pi i}\int_\gamma R_{M,a^+,-a}(z)\, M\varphi\, dz\right] \\
&\qquad = -\frac{1}{2\pi i}\int_\gamma (zR_{M,a^+,-a}(z))\, M\varphi\, dz.
\end{aligned}$$

Proof. Let $\varphi \in A_{a^+}$ and $z \in C^{++}$. Then for all $f \in \mathscr{D}(H_{M,a^-})$

$$\begin{aligned}
&\langle H_{M,a^-} f, R_{M,a^+,-a}(z)\, M\varphi \rangle_{a^-,-a} \\
&\quad = (H_M f, R_M(z)\, M\varphi))_{\mathscr{H}_M} \\
&\quad = (f, (I + zR_M(z))\, M\varphi)_{\mathscr{H}_M} \\
&\quad = \langle f, (I + zR_{M,a^+,-a}(z))\, M\varphi \rangle_{a^-,-a}.
\end{aligned}$$

By analytic continuation in z from $z \in C^{++}$ to $z \in C_a \backslash R_a$ we get for all $f \in \mathscr{D}(H_{M,a^-})$

$$\begin{aligned}
&\langle H_{M,a^-} f, R_{M,a^+,-a}(z)\, M\varphi \rangle_{a^-,-a} \\
&\quad = \langle f, (I + zR_{M,a^+,-a}(z))\, M\varphi \rangle_{a^-,-a}.
\end{aligned} \tag{11}$$

Hence, $R_{M,a^+,-a}(z)\, M\varphi \in \mathscr{D}(H_{M,-a})$ and $(H_{M,-a} - z)\, R_{M,a^+,-a}(z) M\varphi = M\varphi$.

We integrate (11) and get for all $f \in \mathscr{D}(H_{M,a^-})$

$$\begin{aligned}
&\left\langle H_{M,a^-} f, -\frac{1}{2\pi i} \int_\gamma R_{M,a^+,-a}(z)\, M\varphi\, dz \right\rangle_{a^-,-a} \\
&\quad = \left\langle f, -\frac{1}{2\pi i} \int_\gamma (zR_{M,a^+,-a}(z))\, M\varphi\, dz \right\rangle_{a^-,-a}.
\end{aligned}$$

Hence,

$$-\frac{1}{2\pi i} \int_\gamma R_{M,a^+,-a}(z)\, M\varphi\, dz \in \mathscr{D}(H_{M,-a}),$$

and

$$\begin{aligned}
&H_{M,-a}\left[-\frac{1}{2\pi i} \int_\gamma R_{M,a^+,-a}(z)\, M\varphi\, dz \right] \\
&\quad = -\frac{1}{2\pi i} \int_\gamma (zR_{M,a^+,-a}(z))\, M\varphi\, dz.
\end{aligned}$$

The Lemma is proved.

LEMMA 10. *For $\psi(\theta)$ and φ as in (6), we have:*

(1) *Unless $[M\psi]^{\sim}(\theta) \equiv 0$ for $\theta \in S_{a^+}$*

$$[M\psi]^{\sim}(\theta) \notin \mathscr{H}_M \qquad \textit{for} \quad \theta \in \{\theta_1 \mid 0 \leq \operatorname{Im} \theta_1 < -\tfrac{1}{2} \operatorname{Arg} z_0\};$$

explicitly

$$\int_0^\infty \| [M\psi]^\sim(\theta)(\lambda)\|_{\mathscr{A}}^2 \, d\lambda = \int_0^\infty \| [M\psi]^\sim(0)(\lambda)\|_{\mathscr{A}}^2 \, e^{-2\,\mathrm{Im}\,\theta\lambda} \, d\lambda = \infty$$

for $\theta \in \{\theta_1 \mid 0 \leqslant \mathrm{Im}\,\theta_1 < -\frac{1}{2}\,\mathrm{Arg}\, z_0\}$.

(2) $[M\psi]^\sim(0) = -(1/2\pi i)\int_\gamma R_{M,a^+,-a}(z)\, M\varphi\, dz$, *and the residue of* $R_{M,a^+,-a}(z)$ *at* $z = z_0$, *denoted by* $\mathrm{Res}(z_0)$, *is a finite-rank operator.*

(3) *Every function* $g \in \mathscr{R}(\mathrm{Res}(z_0)) \cap N(H_{M,-a} - z_0)$ *has the form* $g = [M\psi]^\sim(0)$, *where* $[M\psi]^\sim(\theta)$ *for* $\theta \in S_{a^+}$ *is given by*

$$[M\psi]^\sim(\theta) = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} MR(ia, z)\, \varphi(ia)\, dz,$$

for some $\varphi \in A_{a^+}$. *Moreover* $[M\psi]^\sim(\theta)$ *above belongs to* $N(H_M(\theta) - z_0)$ *for* $\theta \in S_{z_0}$.

(4) *For fixed* $\theta \in S_{z_0}$, *an isomorphism is defined from* $N(H_M(\theta) - z_0)$ *onto* $\mathscr{R}(\mathrm{Res}(z_0)) \cap N(H_{M,-a} - z_0)$ *given by multiplication with* $e^{-i\theta(\cdot)}$.

Proof. (1) From (8) we get

$$[M\psi]^\sim(\theta) = -\frac{1}{2\pi i} e^{i(\theta - ia)(\cdot)} \int_\gamma MR(ia, z)\, \varphi(ia)\, dz, \qquad \text{for} \quad \theta \in S_{a^+}.$$

Clearly, if $[M\psi]^\sim(\theta) \not\equiv 0$ for $\theta \in S_{a^+}$, then $[M\psi]^\sim(\theta) \neq 0$ for all $\theta \in S_{a^+}$.

Supppose $[M\psi]^\sim(\theta_0) \in \mathscr{H}_M$ for some $\theta_0 \in \{\theta_1 \mid 0 \leqslant \mathrm{Im}\,\theta_1 < -\frac{1}{2}\,\mathrm{Arg}\, z_0\}$. We find a contradiction.

From Lemma 7 and Lemma 8 (2) we get that $[M\psi]^\sim(\theta_0) \in \mathscr{D}(H_M(\theta_0))$ and $(H_M(\theta_0) - z_0)[M\psi]^\sim(\theta_0) = 0$.

Hence $N(H_M(\theta_0) - z_0) \neq 0$ for some θ_0, $0 \leqslant \mathrm{Im}\,\theta_0 < -\frac{1}{2}\,\mathrm{Arg}\, z_0$. But this is not true.

We have a contradiction, and the first part of (1) is proved.

The second part of (1) follows easily from the first part.

(2) For $z \in \gamma$

$$\begin{aligned} MR(ia, z)\, \varphi(ia) &= MR(ia, z)\, M^{-1} M\varphi(ia) \\ &= R_M(ia, z)\, e^{-a(\cdot)} M\varphi. \end{aligned}$$

(Because $M\varphi(\theta) = e^{i\theta(\cdot)} M\varphi$, for $\theta \in \mathbb{R}$, we find by analytic extension in θ to $\theta = ia$ that $M\varphi(ia) = e^{-a(\cdot)} M\varphi$.)

Hence

$$[M\psi]^{\sim}(0) = -\frac{1}{2\pi i} e^{a(\cdot)} \int_\gamma MR(ia, z)\, \varphi(ia)\, dz$$

$$= \frac{-1}{2\pi i} \int_\gamma e^{a(\cdot)} R_M(ia, z)\, e^{-a(\cdot)} M\varphi\, dz$$

$$= -\frac{1}{2\pi i} \int_\gamma R_{M,a^+,-a}(z)\, M\varphi\, dz \qquad \text{(by (3))}.$$

$\mathrm{Res}(z_0)$ is a finite-rank operator, because the residue of $R_M(ia, z)$ at $z = z_0$ is.

(3) Let $g \in \mathscr{R}(\mathrm{Res}(z_0)) \cap N(H_{M,-a} - z_0)$ be given. Then for some $\varphi \in A_{a^+}$

$$g = -\frac{1}{2\pi i} \int_\gamma R_{M,a^+,-a}(z)\, M\varphi\, dz.$$

By using Lemma 9, we find that

$$(H_{M,-a} - z_0)\, g = -\frac{1}{2\pi i} \int_\gamma (z - z_0)\, R_{M,a^+,-a}(z)\, M\varphi\, dz$$

$$= -\frac{1}{2\pi i} e^{a(\cdot)} \int_\gamma (z - z_0)\, R_M(ia, z)\, e^{-a(\cdot)} M\varphi\, dz.$$

Because $(H_{M,-a} - z_0)\, g = 0$ (by assumption) we have

$$-\frac{1}{2\pi i} \int_\gamma (z - z_0)\, R_M(ia, z)\, M\varphi(ia)\, dz = 0.$$

Hence,

$$-\frac{1}{2\pi i} \int_\gamma R_M(ia, z)\, M\varphi(ia)\, dz \in N(H_M(ia) - z_0).$$

We now easily find that

$$[M\psi]^{\sim}(\theta) := -\frac{1}{2\pi i} \int_\gamma e^{i(\theta - ia)(\cdot)}$$

$$\times MR(ia, z)\, \varphi(ia)\, dz \in N(H_M(\theta) - z_0),$$

for $\theta \in S_{z_0}$. Clearly $[M\psi]^{\sim}(0) = g$.

(4) Let $M\psi(\theta) \in N(H_M(\theta) - z_0)$ (fixed $\theta \in S_{z_0}$). Then for some $\varphi \in A_{a^+}$

$$M\psi(\theta) = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} MR(ia, z)\,\varphi(ia)\,dz.$$

By Lemma 10 (2) and Lemma 8 (2), we have that

$$e^{-i\theta(\cdot)} M\psi(\theta) \in \mathscr{R}(\mathrm{Res}(z_0)) \cap N(H_{M,-a} - z_0).$$

By Lemma 10 (3) the map (multiplication with $e^{-i\theta(\cdot)}$) is onto, and clearly it one-to-one.

The Lemma is proved.

Remark. Using Corollary 2.7 in [5] we find that $[M\psi]^{\sim} \notin \mathscr{H}_M$ for $\theta \in \{\theta_1 \mid 0 \leqslant \operatorname{Im}\theta_1 \leqslant -\frac{1}{2}\operatorname{Arg} z\}$ ($[M\psi]^{\sim}(\theta)$ in Lemma 10 (1)), if certain further conditions are assumed on V.

DEFINITION 11. By resonance eigenfunctions at z_0, we mean the functions belonging to $\mathscr{R}(\mathrm{Res}(z_0)) \cap N(H_{M,-a} - z_0)$, where $\mathrm{Res}(z_0)$ is the residue of $R_{M,a^+,-a}(z)$ at $z = z_0$.

We now state our main result. z_0 and γ are as above (Diagram 4).

THEOREM 12. (1) *A finite number of functions, each one given for some* $M\varphi \in \mathscr{H}_{M,a^+}$ *by*

$$M\psi(\theta) = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} R_M(ia, z)\, e^{-a(\cdot)} M\varphi\, dz,$$

form an analytic (in θ) basis for $N(H_M(\theta) - z_0)$ *for*

$$\theta \in S_{z_0} = \{\theta_1 \mid -\tfrac{1}{2}\operatorname{Arg} z_0 < \operatorname{Im}\theta_1 \leqslant a\}.$$

(2) *The basis function $M\psi(\theta)$ defined for $\theta \in S_{z_0}$ in (1) has a $\mathscr{H}_{M,-a}$-valued continuation in θ to $\theta \in S_{a^+}$ given by*

$$[M\psi]^{\sim}(\theta) = -\frac{1}{2\pi i}\int_\gamma e^{i(\theta - ia)(\cdot)} R_M(ia, z)\, e^{-a(\cdot)} M\varphi.$$

(3) $H_{M,-a\,\|\,O(\mathscr{H}_M)} = H_M$.

(4) *For $z \in C^{++}$, $R_M(z)_{\|\,B(\mathscr{H}_{M,a^+}, \mathscr{H}_{M,-a})}$ has a meromorphic continuation in z to C_a, given by*

$$R_{M,a^+,-a}(z) = e^{a(\cdot)} R_M(ia, z)\, e^{-a(\cdot)}.$$

Furthermore $\mathscr{R}(R_{M,a^+,-a}(z))$ and $\mathscr{R}(\mathrm{Res}(z_0))$ are subspaces of $\mathscr{D}(H_{M,-a})$.

(5) *An isomorphism of* $N(H_M(ia)-z_0)$ *onto the space of resonance eigenfunctions at* z_0 *is given by multiplication with* $e^{a(\cdot)}$ *or, equivalently, by analytic continuation as in* (2) *and restriction to* $\theta=0$.

Moreover, fixing a resonance eigenfunction $[M\psi]^{\sim}(0)$, *we have unless* $[M\psi]^{\sim}(0)=0$ *that*

$$\int_0^\infty \| [M\psi]^{\sim}(0)(\lambda)\|_{\mathscr{k}}^2 \, e^{-\operatorname{Im}\theta 2\lambda} \, d\lambda = \infty$$

for all $\theta \in \{\theta_1 \mid 0 \leqslant \operatorname{Im}\theta_1 < -\frac{1}{2}\operatorname{Arg} z_0\}$. *For* $\theta \in S_{z_0}$, $e^{i\theta(\cdot)}[M\psi]^{\sim}(0) \in N(H_M(\theta)-z_0)$.

COROLLARY 13. *Fixing the basis function* $M\psi(\theta)$ *defined for* $\theta \in S_{z_0}$ *in* (1) *we have that the* $\mathscr{H}$*-valued function* $\psi(\theta)=M^{-1}(M\psi(\theta))$ *is maximum analytic in* S_{z_0} *in the following sense*:

For all $\varepsilon>0$, $\psi(\theta)$ *does not have an analytic continuation from* S_{z_0} *to* $\{\theta_1 \mid -\frac{1}{2}\operatorname{Arg} z_0 - \varepsilon < \operatorname{Im}\theta_1 \leqslant a\}$ *as an* $\mathscr{H}$*-valued function.*

2. THE SPECTRUM OF $H_{M,-a}$

LEMMA 14. *For* $z \in \{z_1 \neq 0 \mid 0 < \operatorname{Arg} z_1 \leqslant 2a\}$

$$(H_{0_M}-z)^{-1}_{\|O(\mathscr{H}_{M,a^-})} \notin B(\mathscr{H}_{M,a^-}).$$

Proof. We suppose $(H_{0_M}-z)^{-1}_{\|O(\mathscr{H}_{M,a^-})} \in B(\mathscr{H}_{M,a^-})$ and we find a contradiction.

Using the fact that for $\theta \in \mathbb{R}$, $FU(\theta)=U(-\theta)F$ (F is the Fourier-transform on $\mathscr{H}$), we find easily that given $\varphi \in A_{a^-}$, then

$$F\varphi \in A_{a^+} \qquad \text{and} \qquad \|\varphi(\theta)\| = \|(F\varphi)(-\theta)\| \qquad \text{for} \quad \theta \in S_{a^-}. \quad (12)$$

Remarking that φ defined by $\varphi(r)=e^{-r}$ belongs to A_{a^-}, we have by assumption that $(H_0-z)^{-1}\varphi \in A_{a^-}$.

Let $\theta \in S_{a_-}\backslash\{\theta_1 \mid \operatorname{Im}\theta_1 = -\frac{1}{2}\operatorname{Arg} z\}$, then

$$\begin{aligned}
&\|(H_0-z)^{-1}\varphi\|_{A_{a^-}} \\
&\quad \geqslant \|((H_0-z)^{-1}\varphi)(\theta)\| = \|F(H_0-z)^{-1}\varphi)(-\theta)\| \qquad \text{(by (12))} \\
&\quad = \left\| \left(\frac{1}{(\cdot)^2-z} F\varphi\right)(-\theta) \right\|.
\end{aligned}$$

Because there exists $C \neq 0$, such that $F\varphi = C/((\cdot)^2+1)^2$, we find that

$$\left\| \left(\frac{1}{(\cdot)^2-z} C \left(\frac{1}{(\cdot)^2+1} \right)^2 \right)(-\theta) \right\|$$

is uniformly bounded in $\theta \in S_{a^-}\backslash\{\theta_1 \mid \operatorname{Im}\theta_1 = -\frac{1}{2}\operatorname{Arg} z\}$.

But

$$\left\|\left(\frac{1}{(\cdot)^2-z}\,C\left(\frac{1}{(\cdot)^2+1}\right)^2\right)(-\theta)\right\|^2$$
$$=\int_{\mathbb{R}^3}\left|\frac{1}{e^{-2\theta}k^2-z}\,C\left(\frac{1}{e^{-2\theta}k^2+1}\right)^2\right.$$
$$\left.\times e^{-3/2\theta}\right|^2 dk \to \infty \qquad \text{for} \quad \theta \to -\frac{i}{2}\operatorname{Arg} z.$$

Hence we have found a contradiction, and we have proved that $(H_{0_M}-z)^{-1}_{\| O(\mathscr{H}_{M,a^-})} \notin B(\mathscr{H}_{M,a^-})$.

Letting $\sigma_p(H_M)$ be the set of finite-dimensional, isolated eigenvalues of H_M, we shall prove

THEOREM 15. $\sigma(H_{M,-a})=\sigma_p(H_M)\cup\{z\neq 0 \mid 0\geqslant \operatorname{Arg} z\geqslant -2a\}\cup\{0\}$.

Proof. Remarking that H_{M,a^-} is closed (this is easy to see, because

$$\Gamma(H_{M,a^-})=\Gamma(H_M)\cap\mathscr{H}_{M,a^-}\times\mathscr{H}_{M,a^-} \qquad \text{and} \qquad \|\cdot\|_{\mathscr{H}_{M,a^-}}\geqslant\|\cdot\|_{\mathscr{H}_M}),$$

we have

$$\sigma(H_{M,-a})=\{z \mid \bar{z}\in\sigma(H_{M,a^-})\}.$$

Hence that is enough to prove that $\sigma(H_{M,a^-})$ is equal to $T:=\sigma_p(H_M)\cup\{z\neq 0 \mid 0\leqslant \operatorname{Arg} z\leqslant 2a\}\cup\{0\}$.

Clearly, $\mathbb{R}_-\cap\sigma_p(H_M)\subset\sigma(H_{M,a^-})$. (Remark that every eigenfunction associated with a negative eigenvalue of H_M belongs to $\mathscr{D}(H_{M,a^-})$. This is easy to prove.)

We now prove that $\sigma(H_{M,a^-})\subset T$.

Let $z\notin T$ be given. Then, because $R_M(z)\,\mathscr{H}_{M,a^-}\subset\mathscr{H}_{M,a^-}$

$$(H_{M,a^-}-z)\,R_M(z)_{\| O(\mathscr{H}_{M,a^-})}=I_{\mathscr{H}_{M,a^-}}$$

and

$$R_M(z)_{\| O(\mathscr{H}_{M,a^-})}(H_{M,a^-}-z)=I_{\mathscr{D}(H_{M,a^-})}.$$

Furthermore for all $\varphi\in A_{a^-}$

$$\|R_M(z)_{\| O(\mathscr{H}_{M,a^-})}M\varphi\|^2_{\mathscr{H}_{M,a^-}}$$
$$=\|(R(z)\varphi)(-ia)\|^2+\|R(z)\varphi\|^2$$
$$\leqslant\max\{\|R(-ia,z)\|^2_{B(\mathscr{H})},\|R(z)\|^2_{B(\mathscr{H})}\}\ \|M\varphi\|^2_{\mathscr{H}_{M,a^-}}.$$

Thus $R_M(z)_{\| O(\mathscr{H}_{M,a^-})} \in B(\mathscr{H}_{M,a^-})$, and we have proved that $z \notin \sigma(H_{M,a^-})$. That is, $\sigma(H_{M,a^-}) \subset T$.

To finish the proof, it is enough to show that $\{z \neq 0 \mid 0 < \operatorname{Arg} z \leqslant 2a\} \subset \sigma(H_{M,a^-})$ (remark that $\sigma(H_{M,a^-})$ is closed):

Let $z \in \{z_1 \neq 0 \mid 0 < \operatorname{Arg} z_1 \leqslant 2a\}$ be given and suppose $z \notin \sigma(H_{M,a^-})$. We find a contradiction.

Denoting $(H_{M,a^-} - y)^{-1}$ by $R_{M,a^-}(y)$ for $y \notin \sigma(H_{M,a^-})$, we have by definition

$$(H_{M,a^-} z)\, R_{M,a^-}(z) = I_{\mathscr{H}_{M,a^-}}.$$

Hence

$$R_{M,a^-}(z) = R_M(z)(H_{M,a^-} - z)\, R_{M,a^-}(z) = R_M(z)\, I_{\mathscr{H}_{M,a^-}},$$

that is,

$$R_M(z)_{\| O(\mathscr{H}_{M,a^-})} = R_{M,a^-}(z). \tag{13}$$

We now show that $V_M R_{M,a^-}(z) \in C(\mathscr{H}_{M,a^-})$. Clearly $V_M R_{M,a^-}(z) \in O(\mathscr{H}_{M,a^-})$.

Fix $z_1 \in \mathbb{R}_- \setminus \sigma(H_M)$. Because of the first resolvent equation, $V_M R_{M,a^-}(z) \in C(\mathscr{H}_{M,a^-})$ if $V_M R_{M,a^-}(z_1) \in C(\mathscr{H}_{M,a^-})$; but this is easy to prove [8, p. 219, the proof of Lemma 4.1] and hence $V_M R_{M,a^-}(z) \in C(\mathscr{H}_{M,a^-})$.

Clearly $N(I - V_M R_{M,a^-}(z)) = 0$ (otherwise, by (13) $N(I - V_M R_M(z)) \neq 0$, which is not true).

We now conclude that $[I - V_M R_{M,a^-}(z)]^{-1}$ exists (we use $V_M R_{M,a^-}(z) \in C(\mathscr{H}_{M,a^-})$), and from

$$(H_M - z)^{-1} = (H_{0_M} - z)^{-1}[I - V_M(H_M - z)^{-1}]$$

we find using (13) that

$$(H_{0_M} - z)^{-1}_{\| O(\mathscr{H}_{M,a^-})} = R_{M,a^-}(z)[I - V_M R_{M,a^-}(z)]^{-1} \in B(\mathscr{H}_{M,a^-}). \tag{14}$$

Equation (14) contradicts Lemma 14, and we have proved the theorem.

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